# A zero-one law for first-order logic on random images

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**ABSTRACT:** *For a $n{\times}n$ random image with independent pixels, black with probability $p(n)$ and white with probability $1-p(n)$, the probability of satisfying any given first-order sentence tends to 0 or 1, provided both $p(n)n^{\frac{2}{k}}$ and $(1 - p(n))n^{\frac{2}{k}}$ tend to 0 or $+\infty$, for any integer $k$. The result is proved by computing the threshold function for basic local sentences, and applying Gaifman's theorem.*

## 1 Introduction

The motivation for this work came for the Gestalt theory of vision (see [3] and references therein), a basic idea of which is that the human eye focuses first on remarkable or unusual features of an image, i.e. features that would have a low probability of occurring if the image were random. Hence the natural question: which properties of a random image have a low or high probability? Here we shall deal with the simplest model for random images:

**Definition 1.1** *Let $n$ be a positive integer. Consider the set $X_n = \{1, \ldots, n\}^2$, called the* pixel set. *An* image of size $n{\times}n$ is a mapping from $X_n$ to $\{0, 1\}$ (white or black). Their set is denoted by $E_n$. It is endowed with the product of $n^2$ independent copies of the Bernoulli distribution with parameter $p$, that will be denoted by $\mu_{n,p}$ :

$$\forall \eta \in E_n , \quad \mu_{n,p}(\eta) = \prod_{i,j=1}^{n} p^{\eta(i,j)}(1 - p)^{1-\eta(i,j)} .$$

*A* random image of size $n{\times}n$ and level $p$, denoted by $\mathcal{I}_{n,p}$, is a random element of $E_n$ with distribution $\mu_{n,p}$.

In other words, a random image of size $n \times n$ and level $p$ is a square image in which all pixels are independent, each being black with probability $p$ or white with probability $1-p$.

We shall use the elementary definitions and concepts of first-order logic on finite models, such as described for instance in Ebbinghaus and Flum [4]. Gaifman's theorem ([8] and [4] p. 31) shows that first-order sentences are essentially local. They can be logically reduced to the appearance of fixed subimages (precise definitions will be given in section 2). Assume $p$ is fixed. Then as $n$ tends to infinity, any given subimage of fixed size should appear somewhere in the random image $\mathcal{I}_{n,p}$, with probability tending to 1: this is the two dimensional version of the well known "typing monkey" paradox. It justifies intuitively that the zero-one law should hold for fixed values of $p$. Our main result is more general.

**Theorem 1.2** *Let $p(n)$ be a function from $\mathbb{N}$ into $[0, 1]$ such that:*

$$\forall k = 1, 2, \ldots, \quad \lim_{n\to\infty} n^{\frac{2}{k}} p(n) = 0 \ or \ +\infty \ and \ \lim_{n\to\infty} n^{\frac{2}{k}}(1 - p(n)) = 0 \ or \ +\infty .$$

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*Let A be a first-order sentence. Then:*

$$\lim_{n \to \infty} Prob[\mathcal{I}_{n,p} \models A] = 0 \ or \ 1 \ .$$

Zero-one laws have a long history (cf. Compton [2] for a review and chapter 3 of [4]). The first of them was proved independently by Glebskii et al. [9] and Fagin [6]. It applied to the first-order logic on a finite universe without constraints, and uniform probability. As an example, interpret the elements of $E_n$ as directed graphs with vertex set $\{1, \dots, n\}$, by putting an edge between $i$ and $j$ if pixel $(i, j)$ is black. Then $\mathcal{I}_{n,p}$ becomes a random directed graph (or digraph) with edge probability $p$ (see for instance [11, 12], or [1] for a general reference). As a particular case of the Glebskii et al. – Fagin theorem, the zero-one law holds for first-order propositions on random digraphs. However, first-order logic on images is more expressive than on digraphs, since the geometry of images is not preserved in the graph interpretation.

The theory of random (undirected) graphs was inaugurated by Erdös and Rényi [5] (see [1, 16] for general references). The zero-one law holds for random graphs with edge probability $p$, as a consequence of Oberschelp's theorem [13] on parametric classes (see [4] p. 74 or [16] p. 318). At first, zero-one laws were essentially combinatorial, as they applied to the uniform probability on the set of all structures, corresponding to edge probability $p = \frac{1}{2}$ in the case of graphs. It was soon noticed that they also hold for any fixed value of $p$. But it is well known that random graphs become more interesting by letting $p = p(n)$ tend to 0 as $n$ tends to infinity. A crucial notion for random graphs is the appearance of given subgraphs ([16] p. 309). The threshold function for the appearance of a given subgraph in a random graph is $p(n) = n^{-\frac{v}{e}}$, where $v$ and $e$ are integers. For $p(n) = n^{-\frac{v}{e}}$, the probability of appearance for certain subgraphs does not tend to 0 or 1. Using the extension technique, ([7, 6] and [4] p. 73), Shelah and Spencer [15] made a complete study of those functions $p(n)$ for which the zero-one law holds for random graphs, and proved in particular that it does for $p(n) = n^{-\alpha}$, for any irrational $\alpha$. Theorem 1.2 is the analogue for random images of Shelah and Spencer's result. To understand why, first notice that the random image model is invariant through exchanging black and white, together with $p$ and $1-p$. Thus we will consider only functions $p(n)$ tending to 0. We shall define precisely the notion of threshold function in section 3, and prove that all threshold functions for patterns are of type $p(n) = n^{-\frac{2}{k}}$: the zero-one law does not hold for these values. For instance, if $p(n)$ is small (resp.: large) compared to $n^{-2}$, the probability of having at least one black pixel tends to 0 (resp.: 1). But for $p(n) = n^{-2}$, it tends to $1 - e^{-1}$. Theorem 1.2 essentially says that the zero-one law holds for any function $p(n)$ which is not a threshold function.

It is worth pointing out here that theorem 1.2 can be extended to other random structures, along two different directions. Firstly, we chose to restrict the study to binary images, using a single unary relation in the language (cf. section 2). With slight modifications of the proofs, and the values of threshold functions, one could introduce a finite set of "color" unary relations, allowing for the coding of multilevel gray or color images. The other possible generalisation concerns the type of graphs. An image is essentially a colored square lattice. The crucial property of that graph for our proof is that there exists a fixed number of vertices at fixed distance of any vertex (balls have bounded cardinality). Our study extends to any family of graphs with bounded balls. For instance, theorem 1.2 also holds for a randomly colored $d$-dimensional square lattice with $n^d$ points, up to replacing $n^{\frac{2}{k}}$

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by $n^{\frac{d}{k}}$ in its statement.

Section 2 is devoted to first-order logic on images. There we shall discuss basic local sentences (definition 2.2 and [4] p. 31), and reduce them to combinations of "pattern sentences" (definition 2.3), showing that a zero-one law holds for all first-order sentences if it holds for basic local or pattern sentences (proposition 2.4). This will trivially imply that theorem 1.2 holds for fixed values of $p$. The section will end with two examples of (second-order) sentences whose probability under $\mu_{n,\frac{1}{2}}$ tends to $\frac{1}{2}$.

In section 3, we shall define the notion of threshold function (definition 3.2) and prove that all threshold functions for basic local sentences are of type $n^{-\frac{2}{k}}$ (proposition 3.4). Theorem 1.2 easily follows from propositions 2.4 and 3.4.

## 2   First-order logic for images

We shall follow the notations and definitions in chapter 0 of [4] for the syntax and semantics of first-order logic. The *vocabulary* is the set of *relations* (or predicates). They apply to the *universe* (or domain). In our case the universe will be the pixel set $X_n$. Image properties will not only be statements on colors of pixels but also about their geometrical arrangement. Our vocabulary will consist of 1 unary and 4 binary relations. The unary relation $C$ is interpreted as the color: $Cx$ means that $x$ is a black pixel and $\neg Cx$ that it is white. Before defining the binary relations, the geometry of $X_n$ needs to be precised.

The pixel set $X_n$ is embedded in $\mathbb{Z}^2$, and naturally endowed with a *graph structure*. In image analysis (see for instance chapter 6 of Serra [14]), the cases most often considered are:

- the 4-*connectivity*. For $i, j > 0$, the neighbors of $(i, j)$ are:

$$(i + 1, j), (i - 1, j), (i, j + 1), (i, j - 1) .$$

- the 8-*connectivity*. The 4 diagonal neighbors are also included:

$$(i + 1, j + 1), (i - 1, j + 1), (i + 1, j - 1), (i - 1, j - 1) .$$

At this point a few words about the borders are needed. In order to avoid particular cases (pixels having less than 4 or 8 neighbors), we shall impose a periodic boundary, deciding for instance that $(1, j)$ is neighbor with $(n, j), (n, j - 1)$, and $(n, j + 1)$, so that the graph becomes a regular 2-dimensional torus. Although it may seem somewhat unnatural for images, without that assumption the zero-one law would fail. Consider indeed the (first-order) sentence "there exist 4 black pixels each having only one horizontal neighbor". Without periodic boundary conditions, it applies to the 4 corners, and the probability for a random image $\mathcal{I}_{n,p}$ to satisfy it is $p^4$. From now on, the identification $n + 1 \equiv 1$ holds for all operations on pixels.

Once the graph structure is fixed, the relative positions of pixels can be described by binary predicates. In the case of 4-connectivity 2 binary predicates suffice, $U$ (up) and $R$ (right): $Uxy$ means that $y = x + (0, 1)$ and $Rxy$ that $y = x + (1, 0)$. In the case of 8-connectivity, two more predicates must be added, $D_1$ and $D_2$: $D_1xy$ means that $y = x + (1, 1)$ and $D_2xy$ that $y = x + (1, -1)$. For convenience reasons, we shall stick to 8-connectivity. Thus the vocabulary of images is the set $\{C, U, R, D_1, D_2\}$. Once the universe and the vocabulary are fixed,

the *structures* are particular models of the relations, applied to variables in the domain. To any structure, a graph is naturally associated ([4] p. 26), connecting those pairs of elements $\{x, y\}$ which are such that $Sxy$ or $Syx$ are satisfied, where $S$ is any of the binary relations. Of course only those structures for which the associated graph is the square lattice with diagonals and periodic boundaries will be called images. As usual, the *graph distance $d$* is defined as the minimal length of a path between two pixels. We shall denote by $B(x, r)$ the ball of center $x$ and radius $r$:

$$B(x, r) = \{\, y \in X_n \,;\, d(x, y) \leq r \,\}$$

In the case of 8-connectivity, $B(x, r)$ is a square containing $(2r + 1)^2$ pixels.

Formulas such as $Cx$, $Uxy$, $Rxy$... are called *atoms*. The *first-order logic* ([4] p. 5) is the set of all formulas obtained by recursively combining first-order formulas, starting with atoms.

**Definition 2.1** *The set $\mathcal{L}_1$ of first-order formulas is defined by:*

1. *All atoms belong to $\mathcal{L}_1$.*

2. *If $A$ and $B$ are first-order formulas, then $(\neg A)$, $(\forall x A x)$ and $(A \wedge B)$ also belong to $\mathcal{L}_1$.*

Here are two examples of first-order formulas:

1. $\forall x, y,\ (Rxy \wedge Uyz) \to D_1 xz$,

2. $(\exists y\ (Rxy \wedge Uyz)) \leftrightarrow D_1 xz$

Notice that any image satisfies them both: adding the two diagonal relations $D_1$ and $D_2$ does not make the language any more expressive. The only reason why the 8-connectivity was preferred here is that the corresponding balls are squares.

We are interested in formulas for which it can be decided if they are true or false for any given image, i.e. for which all variables are quantified. They are called *closed formulas*, or *sentences*. Such a sentence $A$ defines a subset $A_n$ of $E_n$: that of all images $\eta$ that *satisfy $A$* ($\eta \models A$). Its probability for $\mu_{n,p}$ will still be denoted by $\mu_{n,p}(A)$.

$$\mu_{n,p}(A) = Prob[\mathcal{I}_{n,p} \models A] = \sum_{\eta \models A} \mu_{n,p}(\eta) \ .$$

Gaifman's theorem ([4] p. 31), states that every first-order sentence is equivalent to a boolean combination of *basic local sentences*.

**Definition 2.2** *A basic local sentence has the form:*

$$\exists x_1 \ldots \exists x_m \left( \bigwedge_{1 \leq i < j \leq m} d(x_i, x_j) > 2r \right) \wedge \left( \bigwedge_{1 \leq i \leq m} \psi_i(x_i) \right) , \qquad (1)$$

*where:*

- *$m$ and $r$ are fixed nonnegative integers,*

- *for all $i = 1, \ldots, m$, $\psi_i(x) \in \mathcal{L}_1$ is a formula for which only variable $x$ is free (not bound by a quantifier), and the other variables all belong to the ball $B(x, r)$.*

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For any $x$ and a fixed radius $r$, consider now a *complete description* $D(x)$ of the ball $B(x,r)$, i.e. a first-order sentence for which all statements concerning pixels at distance at most $r$ of $x$ are either asserted or negated. There exists a single image $I_D$ of size $(2r+1) \times (2r+1)$, centered at $x$, satisfying it. Thus $D(x)$ can be interpreted as: "the pattern of pixels at distance at most $r$ of $x$ is $I_D$".

**Definition 2.3** *A pattern sentence has the form:*

$$\exists x_1 \ldots \exists x_m \left( \bigwedge_{1 \le i < j \le m} d(x_i, x_j) > 2r \right) \wedge \left( \bigwedge_{1 \le i \le m} D_i(x_i) \right), \qquad (2)$$

*where:*

- *$m$ and $r$ are fixed nonnegative integers,*

- *for all $i = 1, \ldots, m$, $D_i(x)$ is a complete description of the ball $B(x,r)$.*

Examples of (interpreted) pattern sentences are:

1. "there exist 3 black pixels",

2. "there exists a $3 \times 3$ white square",

3. "there exist 3 non overlapping $5 \times 5$ white squares with a black pixel on the center".

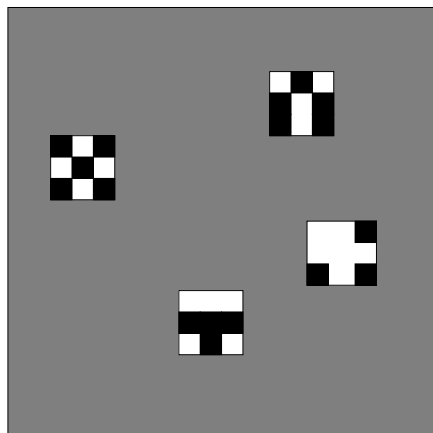Figure 1 gives another illustration. Obviously, pattern sentences are particular



Figure 1: Illustration of a pattern sentence, for $m = 4$ and $r = 1$.

cases of basic local sentences. Proposition 2.4 below reduces the proof of zero-one laws for random images to pattern sentences.

**Proposition 2.4** *Consider the following three assertions.*

*(i) The probability of any* pattern *sentence tends to $0$ or $1$.*

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*(ii)  The probability of any* basic local *sentence tends to* 0 *or* 1.

*(iii)  The probability of any* first order *sentence tends to* 0 *or* 1.

*Then (i) implies (ii) and (ii) implies (iii).*

*Proof*: Observe first that if the probabilities of sentences $A$ and $B$ tend to 0 or 1, then so do the probabilities of $\neg A$ and $A \wedge B$. This follows from elementary properties of probabilities. As a consequence, if the probability of $A$ tends to 0 or 1 for any $A$ in a given family, this remains true for any finite boolean combination of sentences in that family. Thus Gaifman's theorem yields that *(ii)* implies *(iii)*. We shall prove now that every basic local sentence is either unsatisfiable or a finite boolean combination of pattern sentences. Indeed, consider a formula $\psi(x)$ for which only variable $x$ is free, and the other variables all belong to the ball $B(x, r)$. Either it is not satisfiable, or there exists a finite set of $(2r + 1) \times (2r + 1)$ images (at most $2^{(2r+1)^2}$) which satisfy it. To each of those images corresponds a complete description $D(x)$ which implies $\psi(x)$. So $\psi(x)$ is equivalent to the disjunction of these $D(x)$'s:

$$\psi(x) \leftrightarrow \bigvee_{D(x) \to \psi(x)} D(x) \ . \tag{3}$$

In formula (1), one can replace each $\psi_i(x_i)$ by a disjunction of complete descriptions. Rearranging terms, one sees that the basic local sentence (1) is itself a finite disjunction of pattern sentences. $\qquad\square$

The zero-one law for fixed values of $p$ is an easy consequence of proposition 2.4. Indeed, for fixed $p$, the probability of any pattern sentence tends to 1. To see why, consider the following sentence:

$$\exists x \left( \bigwedge_{1 \leq i \leq m} D_i(x + ((i-1)(2r+1), 0)) \right) \ , \tag{4}$$

interpreted as: "subimages $I_{D_1}, \ldots, I_{D_m}$ appear in $m$ consecutive, horizontally adjacent balls of radius $r$". It clearly implies (2). But (4) is equivalent to the appearance of a given subimage on a rectangle of size $(2r+1) \times (m(2r+1))$. This occurs in a random image $\mathcal{I}_{n,p}$ with probability tending to 1 as $n$ tends to infinity. Thus (2) has a probability tending to one of being satisfied by $\mathcal{I}_{n,p}$.

This section ends with two counter-examples of (second-order) sentences the probability of which does not tend to 0 or 1. The first one is "the number of black pixels is even". This is one of the basic examples of second order sentences, that do not belong to first order logic (see [4] example 1.3.4 p. 21 and p. 37). Its probability is

$$\frac{1}{2} \sum_{k=1}^{n^2} \binom{n^2}{k} p^k (1-p)^{n^2-k} (1 + (-1)^k) = \frac{1}{2} (1 + (1-2p)^{n^2}) \ ,$$

which tends to $\frac{1}{2}$ for any $p$ such that $0 < p < 1$.

The second example is more relevant to images. Define a 6-*connected path* as a path where the directions $(-1, 1)$ and $(1, -1)$ are forbidden, or more precisely a $m$-tuple of pixels $(x_1, \ldots, x_m)$, such that for $i = 1, \ldots, m-1$, $x_{i+1} \in x_i \pm \{(1, 0), (0, 1), (1, 1)\}$, and the borders of the image are not crossed (see an illustration on figure 2). Consider now the two sentences:

1. *BLR*: "there exists a 6-connected path of black pixels from left to right",

2. *WTB*: "there exists a 6-connected path of white pixels from top to bottom".

Some geometrical considerations show that an image satisfies *BLR* if and only if it does not satisfy *WTB* (this would not hold for 4- or 8-connected paths: see [14] p. 183). Take now $p = \frac{1}{2}$. Symmetry implies that

$$\mu_{n,\frac{1}{2}}(BLR) = \mu_{n,\frac{1}{2}}(WTB) \ ,$$

hence both probabilities must be equal to $\frac{1}{2}$.

The sentences *BLR* and *WTB* are examples of those properties studied by percolation theory (see Grimmett [10] for a general reference). Actually the random image model that we consider here is a finite approximation of site percolation ([10] p. 24). Using percolation techniques, one can prove that $\mu_{n,p}(BLR)$ tends to 0 if $p < \frac{1}{2}$, to 1 if $p > \frac{1}{2}$.
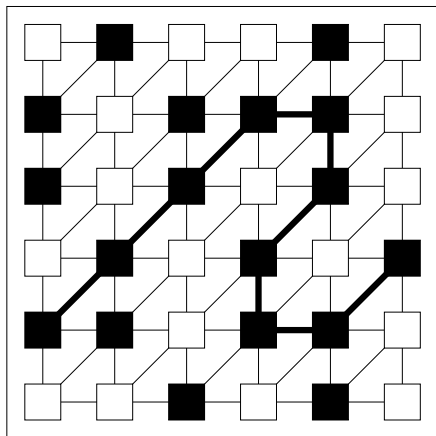


Figure 2: A 6-connected path of black pixels from left to right.

# 3 Threshold functions for basic local sentences

The notions studied in this section have exact counterparts in the theory of random graphs as presented by Spencer [16]. We begin with the asymptotic probability of single pattern sentences, which correspond to the appearance of subgraphs ([16] p. 309).

**Proposition 3.1** *Let $r$ and $k$ be two integers such that $0 < k < (2r + 1)^2$. Let $I$ be a fixed $(2r + 1) \times (2r + 1)$ image, with $k$ black pixels and $h = (2r + 1)^2 - k$ white pixels. Let $D(x)$ be the complete description of the ball $B(x, r)$ satisfied only by a copy of $I$, centered at $x$. Let $\tilde{D}$ be the sentence $(\exists x \ D(x))$. Let $p = p(n)$ be a function from $\mathbb{N}$ to $[0, 1]$.*

$$If \ \lim_{n \to \infty} n^2 p(n)^k = 0 \ then \ \lim_{n \to \infty} \mu_{n,p(n)}(\tilde{D}) = 0 \ . \tag{5}$$

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$$If \lim_{n\to\infty} n^2 p(n)^k (1 - p(n))^h = +\infty \ then \ \lim_{n\to\infty} \mu_{n,p(n)}(\tilde{D}) = 1 \ . \qquad (6)$$

$$If \lim_{n\to\infty} n^2 (1 - p(n))^h = 0 \ then \ \lim_{n\to\infty} \mu_{n,p(n)}(\tilde{D}) = 0 \ . \qquad (7)$$

*Proof*: We already noticed the symmetry of the problem: swapping black and white together with $p$ and $1-p$ should leave statements unchanged. In particular the proofs of (5) and (7) are symmetric, and only the former will be given.

For a given $x$, the probability of occurence of $I$ in the ball $B(x, r)$ is :

$$\mu_{n,p(n)}(D(x)) = p(n)^k (1 - p(n))^h \ .$$

The pattern sentence $\tilde{D}$ is the disjunction of all $D(x)$'s:

$$\tilde{D} \leftrightarrow \bigvee_{x \in X_n} D(x) \ .$$

Hence:

$$\mu_{n,p(n)}(\tilde{D}) \le n^2 p(n)^k (1 - p(n))^h \ ,$$

from which (5) follows.

Consider now the following set of pixels:

$$T_n = \left\{ (r + 1 + \alpha(2r + 1), r + 1 + \beta(2r + 1)) , \ \alpha, \beta = 0, \dots, \lfloor \tfrac{n}{2r+1} \rfloor - 1 \right\} , \qquad (8)$$

where $\lfloor \cdot \rfloor$ denotes the integer part. Call $\tau(n)$ the cardinality of $T(n)$:

$$\tau(n) = \left\lfloor \frac{n}{2r + 1} \right\rfloor^2 \ ,$$

which is of order $n^2$. Notice that the disjunction of $D(x)'s$ for $x \in T_n$ implies $\tilde{D}$.

$$\bigvee_{x \in T_n} D(x) \to \tilde{D} \ .$$

The distance between any two distinct pixels $x, y \in T_n$ is larger than $2r$, and the balls $B(x, r)$ and $B(y, r)$ do not overlap. Therefore the events "$\mathcal{I}_{n,p} \models D(x)$" for $x \in T_n$ are mutually independent. Thus:

$$\begin{aligned}
\mu_{n,p(n)}(\tilde{D}) &\ge \mu_{n,p(n)} \left( \bigvee_{x \in T_n} D(x) \right) \\
&= 1 - \left( 1 - p(n)^k (1 - p(n))^h \right)^{\tau(n)} \\
&\ge 1 - \exp(-\tau(n) p(n)^k (1 - p(n))^h) \ ,
\end{aligned}$$

hence (6). $\qquad\qquad \square$

Due to the symmetry of the model, we shall consider from now on that $p(n) < \frac{1}{2}$. Proposition 3.1 shows that the appearance of a given subimage only depends on its number of black pixels: if $p(n)$ is small compared to $n^{-\frac{2}{k}}$, then no subimage of fixed

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size, with $k$ black pixels, should appear in $\mathcal{I}(n, p(n))$. If $p(n)$ is large compared to $n^{-\frac{2}{k}}$, all subimages with $k$ black pixels should appear. Proposition 3.1 does not cover the particular cases $k = 0$ (appearance of a white square) and $k = (2r+1)^2$ (black square). They are easy to deal with. Denote by $W$ (resp.: $B$) the pattern sentence $(\exists x\, D(x))$, where $D(x)$ denotes the complete description of $B(x, r)$ being all white (resp.: all black). Then $\mu_{n,p(n)}(W)$ always tends to 1 (remember that $p(n) < \frac{1}{2}$). Statements (5) and (6) apply to $B$, with $k = (2r+1)^2$.

The notion of *threshold function* is a formalisation of the behaviors that have just been described.

**Definition 3.2** *Let $A$ be a sentence. A threshold function for $A$ is a function $r(n)$ such that:*

$$\lim_{n \to \infty} \frac{p(n)}{r(n)} = 0 \ \ implies \ \ \lim_{n \to \infty} \mu_{n,p(n)}(A) = 0 \ ,$$

*and :*

$$\lim_{n \to \infty} \frac{p(n)}{r(n)} = +\infty \ \ implies \ \ \lim_{n \to \infty} \mu_{n,p(n)}(A) = 1 \ .$$

Notice that a threshold function is not unique. For instance if $r(n)$ is a threshold function for $A$, then so is $cr(n)$ for any positive constant $c$. It is costumary to ignore this and talk about "the" threshold function of $A$. For instance, the threshold function for "there exists a black pixel" is $n^{-2}$.

Proposition 3.1 essentially says that the threshold function for the appearance of a given subimage $I$ is $n^{-\frac{2}{k}}$, where $k$ is the number of black pixels in $I$. Proposition 3.4 below will show that the threshold function for a basic local sentence $L$ is $n^{-\frac{2}{k(L)}}$, where $k(L)$ is an integer that we call the *index* of $L$. Its definition refers to the decomposition (3) of a local property into a finite disjunction of complete descriptions, already used in the proof of proposition 2.4.

**Definition 3.3** *Let $L$ be the basic local sentence defined by:*

$$\exists x_1 \ldots \exists x_m \ \left( \bigwedge_{1 \le i < j \le m} d(x_i, x_j) > 2r \right) \wedge \left( \bigwedge_{1 \le i \le m} \psi_i(x_i) \right) \ .$$

*If $L$ is not satisfiable, then we shall set $k(L) = +\infty$. If $L$ is satisfiable, for each $i = 1, \ldots, m$, consider the finite set $\{D_{i,1}, \ldots, D_{i,d_i}\}$ of those complete descriptions on the ball $B(x_i, r)$ which imply $\psi_i(x_i)$.*

$$\psi_i(x_i) \leftrightarrow \bigvee_{1 \le j \le d_i} D_{i,j}(x_i) \ .$$

*Each complete description $D_{i,j}(x_i)$ corresponds to an image on $B(x_i, r)$. Denote by $k_{i,j}$ its number of black pixels.*

*The index of $L$, denoted by $k(L)$ is defined by:*

$$k(L) = \max_{i=1}^{m} \min_{j=1}^{d_i} k_{i,j} \ . \tag{9}$$

The intuition behind definition 3.3 is the following. Assume $p(n)$ is small compared to $n^{-\frac{2}{k(L)}}$. Then there exists $i$ such that none of the $D_{i,j}(x_i)$ can be satisfied, therefore there is no $x_i$ such that $\psi_i(x_i)$ is satisfied, and $L$ is not satisfied. On the contrary, if $p(n)$ is large compared to $n^{-\frac{2}{k(L)}}$, then for all $i = 1, \ldots, m$, $\psi_i(x_i)$ should be satisfied for at least one pixel $x_i$, and the probability of satisfying $L$ should be large. In other words, $n^{-\frac{2}{k(L)}}$ is the threshold function of $L$.

**Proposition 3.4** *Let $L$ be a basic local property, and $k(L)$ be its index. If $L$ is satisfiable and $k(L) > 0$, then its threshold function is $n^{-\frac{2}{k(L)}}$. If $k(L) = 0$, its probability tends to 1 (for $p(n) < \frac{1}{2}$).*

*Proof:* Assume $L$ is satisfiable (otherwise its probability is null) and $k(L) > 0$. Let $r(n) = n^{-\frac{2}{k(L)}}$. For $p(n) < \frac{1}{2}$, we need to prove that $\mu_{n,p(n)}$ tends to 0 if $p(n)/r(n)$ tends to 0, and that it tends to 1 if $p(n)/r(n)$ tends to $+\infty$. The former will be proved first.

Consider again the decomposition of $L$ into complete descriptions:

$$L \leftrightarrow \exists x_1 \ldots \exists x_m \left( \bigwedge_{1 \le i < j \le m} d(x_i, x_j) > 2r \right) \wedge \left( \bigwedge_{1 \le i \le m} \bigvee_{1 \le j \le d_i} D_{i,j}(x_i) \right) .$$

If $p(n)/r(n)$ tends to 0, there exists $i$ such that:

$$\forall j = 1, \ldots, d_i , \ \lim_{n \to \infty} n^2 p(n)^{k_{i,j}} = 0 .$$

By proposition 3.1, the probability of $(\exists x \ D_{i,j}(x))$ tends to zero for all $j = 1, \ldots, d_i$. Therefore the probability of $(\exists x \, \psi_i(x))$ tends to 0, which implies that $\mu_{n,p(n)}(L)$ tends to 0.

Conversely, for each $i = 1, \ldots, m$, choose one of the $D_{i,j}(x)$'s, such that the number of black pixels in the corresponding image is minimal (among all $k_{i,j}$'s). Denote that particular description by $D_i(x)$. Consider now the following pattern sentence, which implies $L$:

$$\exists x_1 \ldots \exists x_m \left( \bigwedge_{1 \le i < j \le m} d(x_i, x_j) > 2r \right) \wedge \left( \bigwedge_{1 \le i \le m} D_i(x_i) \right) . \tag{10}$$

As in the proof of proposition 3.1, we shall use the lattice $T_n$, defined by (8). Remember that its cardinality $\tau(n)$ is of order $n^2$. The pattern sentence (10) is implied by:

$$\exists x_1 \ldots \exists x_m \left( \bigwedge_{1 \le i \le m} x_i \in T_n \right) \wedge \left( \bigwedge_{1 \le i < j \le m} x_i \ne x_j \right) \wedge \left( \bigwedge_{1 \le i \le m} D_i(x_i) \right) . \tag{11}$$

Assume first that $k(L) = 0$. Then necessarily, for each $i$, the image corresponding to $D_i(x)$ has only white pixels. With $p(n) < \frac{1}{2}$, the probability of observing a $(2r + 1) \times (2r + 1)$ white image is larger than $\pi = 2^{-(2r+1)^2}$. Since subimages

centered at the points of $T_n$ are independent, the probability of (11) is larger than:

$$1 - \sum_{l=0}^{m-1} \binom{\tau(n)}{l} \pi^l (1-\pi)^{\tau(n)-l} \ ,$$

which tends to 1 as $n$ tends to infinity.

Assume now that $k(L) > 0$. The images corresponding to the minimal descriptions $D_i$ need not be all different: renumber *different* descriptions $D_i$ as $D'_1, \dots, D'_{m'}$. Denote by $k(i)$ the number of black pixels of $D'_i$ (hence $k(L) = \max\{k(i)\}$). Let $\pi_i(n)$ be the probability of $D'_i(x)$, for a given $x$:

$$\pi_i(n) = p(n)^{k(i)} (1 - p(n))^{(2r+1)^2 - k(i)} \ .$$

From the random image $\mathcal{I}_{n,p}$ define the random variable $N_i$ as the number of those pixels $x_i \in T_n$ such that $\mathcal{I}_{n,p}$ is described by $D'_i(x_i)$ on the ball $B(x_i, r)$. Since the different balls do not overlap, $N_i$ has a binomial distribution, with parameters $\tau(n)$ and $\pi_i(n)$. Assuming $p(n)/r(n)$ tends to $+\infty$, it is easy to check that the product $\tau(n)\pi_i(n)$ also tends to infinity. Indeed, $\tau(n)$ is of order $n^2$, and $r(n)^{k(i)} = n^{-2\frac{k(i)}{k(L)}}$ is at least of order $n^{-2}$. Therefore the probability that $N_i$ is larger than $m$ tends to 1 for each $i$, which implies that the probability for all the $N_i$'s to be larger than $m$ also tends to 1. But if all the $N_i$'s are larger than $m$, then one can be sure that all the $D_i(x_i)$ are satisfied for different centers $x_1, \dots, x_m$ of the lattice $\mathcal{T}_n$. Therefore $\mathcal{I}_{n,p}$ satisfies (11), hence (10) and $L$. $\qquad\square$

Having characterized the threshold functions of all basic local properties, the proof of theorem 1.2 is now clear. If $p(n)n^{\frac{2}{k}}$ tends to 0 or $+\infty$ for any positive integer $k$, then by proposition 3.4 the probability of any basic local sentence tends to 0 or 1. This remains true for any boolean combination of basic local sentences (cf. proposition 2.4). By Gaifman's theorem, these boolean combinations cover all first-order sentences. Hence the zero-one law holds for first-order logic.

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