EUROPHYSICS LETTERS 15 December 2005

Europhys. Lett., **72** (6), pp. 997–1003 (2005)
DOI: 10.1209/epl/i2005-10326-5

Percolation fractal exponents without fractals and a new conservation law in diffusion

A. Desolneux[1] and B. Sapoval[2,3]

[1] *Mathématiques Appliquées à Paris 5, CNRS UMR 8145, Université Paris 5 75006 Paris, France*
[2] *Centre de Mathématiques et de leurs Applications, CNRS UMR 8536 École Normale Supérieure - 94235 Cachan, France*
[3] *Laboratoire de Physique de la Matière Condensée, CNRS UMR 7643 École Polytechnique, 91128 Palaiseau, France*

received 14 September 2005; accepted 13 October 2005
published online 11 November 2005

PACS. 64.60.Ak – Renormalization-group, fractal, and percolation studies of phase transitions.
PACS. 05.40.-a – Fluctuation phenomena, random processes, noise, and Brownian motion.
PACS. 64.60.Fr – Equilibrium properties near critical points, critical exponents.

Abstract. – Classically, percolation critical exponents are linked to power laws that characterize cluster fractal properties. We find here that the gradient percolation power laws are conserved even for extreme gradient values for which the frontier of the infinite cluster is no longer fractal. In particular, the exponent 7/4 which was recently shown to be the exact value for the dimension of the so-called "hull" or external perimeter of the incipient percolation cluster, keeps its value in describing the width and length of gradient percolation frontiers whatever the gradient value. Its origin is then not to be found in the thermodynamic limit. The comparison between the numerical and the exact results that can be obtained analytically for extreme values of the gradient suggests that there exists a unique power law from size 2 to infinity that describes the gradient percolation frontier. These results provides an intrinsic method to find whether a rough interface belongs to gradient percolation without knowledge of the gradient and can be considered as resulting from a new conservation law for diffusion on a lattice.

Spreading of objects in space with a gradient of probability is most common. From chemical composition gradients to the distribution of plants that depends on their solar exposure, probability gradients exist in many inhomogeneous systems. In fact inhomogeneity is a rule in nature. However, most of the systems that physicists are studying are homogeneous since they are thought to be simpler to understand. In particular, phase transitions or critical phenomena are studied in this framework, the simplest being percolation transition [1].

In this work, the opposite situation, a strongly inhomogeneous system is studied. We report that percolation fractal exponents are valid even in cases that are far from the large homogeneous system limit. This is found in the frame of gradient percolation, a situation first encountered in the study of diffusion fronts [2, 3]. Surprisingly, these exponents, up to now believed to express properties of large systems, are shown here to be verified in a limit that could be called the small system limit (SSL), where some of the gradient percolation properties can be computed *analytically*. This letter presents the case of the 2D square lattice.

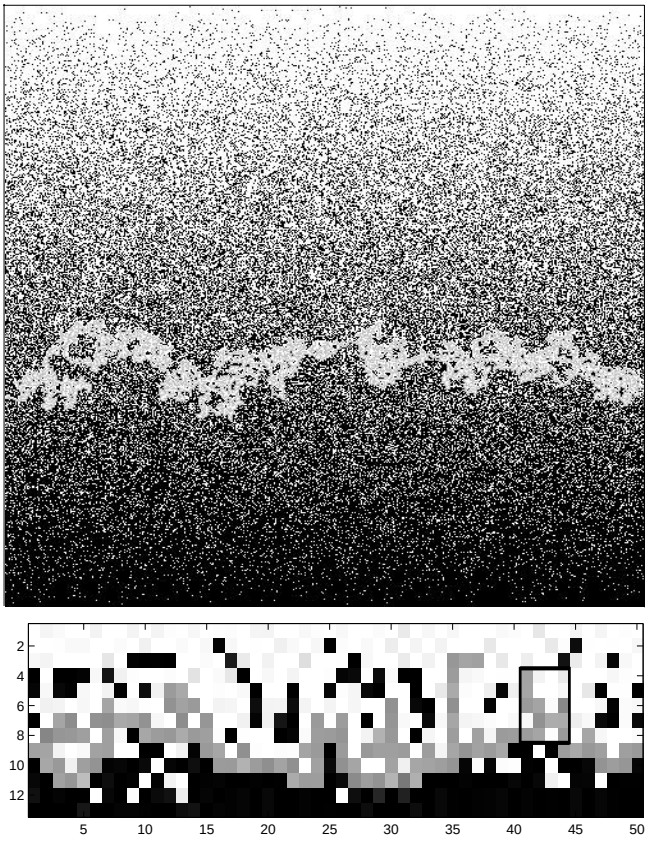

Fig. 1 – Gradient percolation front. Particles are distributed at random with probability $p(x) = 1 - x/L_g$, x being the vertical direction. The occupied sites are in black and the last line of connected occupied sites is the gradient percolation front shown in light gray. This situation corresponds also to a diffusion situation where the front is called the diffusion front. Top: $L = L_g = 500$. Bottom: $L = 50$ and $L_g = 12$. The black window has an horizontal width equal to σ_f. It contains approximately L_g points.

The gradient percolation (GP) situation is depicted in fig. 1. It shows examples of a random distribution of points on a 2D square lattice of size $L_g \times L$ with a linear gradient of concentration. Each site (x, y) is occupied with probability $p(x) = 1 - x/L_g$ (x being the vertical direction in the figure). In gradient percolation there is always an infinite cluster of occupied sites as there is a region where p is larger than the standard percolation (SP) threshold p_c. There is also an infinite cluster of empty sites as there is a region where p is smaller than p_c. The object of interest is the GP front, the external limit (or frontier) of the infinite occupied cluster. It is constituted by the sites which belong to the occupied cluster and are first-nearest neighbours with empty sites belonging to the infinite empty cluster. It is shown in grey in fig. 1. This front is a random object with an average position x_f, a statistical width σ_f and a total length N_f. In so far as the GP front and the SP external perimeter (often called hull) have the same geometry their fractal dimension was first conjectured to be exactly equal to 7/4 in [2]. This result was then demonstrated heuristically by Saleur and Duplantier [4] and very recently it was proved mathematically by Smirnov and Werner [5, 6].

The early GP studies were focussed on finding its relation with standard percolation. Let us first recall the definitions. For $0 \leq x \leq L_g$, $n_f(x)$ is the mean number, per unit horizontal length, of points of the front lying on the line x. It measures the front density at distance x. The length N_f, the position x_f and the width σ_f of the front are then defined in terms of the $n_f(x)$ by

$$N_f = L \sum_{x=0}^{L_g} n_f(x), \qquad x_f = \frac{\sum_{x=0}^{L_g} x n_f(x)}{\sum_{x=0}^{L_g} n_f(x)},$$

$$\text{and } \sigma_f^2 = \frac{\sum_{x=0}^{L_g} (x - x_f)^2 n_f(x)}{\sum_{x=0}^{L_g} n_f(x)}.$$

It was found in [2] that the mean front was situated at a distance where the density of occupation was very close to p_c or $p(x_f) \simeq p_c$. This was verified numerically with such precision that the gradient percolation method is now often used to compute percolation thresholds [7–11]. It was also found that:

1) The width σ_f depends on L_g through a power law $\sigma_f \propto (L_g)^{\nu/(1+\nu)}$, where $\nu = 4/3$ is the correlation length exponent [1] so that $\sigma_f \propto (L_g)^{4/7}$. The width σ_f was also shown to be a percolation correlation length. (We consider the case where L is much larger than the front width).

2) The front was fractal with a dimension D_f, numerically determined, close to 1.75. The front length followed a power law $N_f \propto (L_g)^{\alpha_N}$ with $\alpha_N = (D_f - 1)\nu/(1 + \nu)$.

3) Most important, it was numerically observed that the sum of these two exponents was very close to 1. If true this implies that $\nu/(1 + \nu) + (D_f - 1)\nu/(1 + \nu) = 1$ or $D_f = 1 + 1/\nu = 7/4$. This is how it was conjectured in [2] that $D_f = 7/4$.

In this sense the GP power laws were thought to be linked to the SP exponent ν and to the fractality of the percolation cluster hull. The behavior of the front in the thermodynamic limit has been discussed in [12].

However, if true, and now we know that 7/4 is the exact value [5, 6], there follows an intriguing relation, namely $\sigma_f^{D_f}$ is exactly proportional to L_g^1. This means that the number of surface particles within the correlation length is exactly proportional to L_g. This is particularly striking for diffusion fronts. Diffusion of particles from a source results in a concentration gradient and an associated GP situation. In that frame, the above result means that, if L_g particles have diffused on a vertical row, *there is on average the same, or a constant fraction of, number of particles on the correlated surface.* This surface content of a box with a lateral size equal to the statistical width is illustrated in fig. 1 (bottom). In itself, this fact seems *a priori* to have nothing to do with fractals, percolation or the thermodynamic limit. From this point of view it is possibly the expression of a conservation law. But if such a conservation exists, it should apply also for extreme gradients corresponding to L_g of a few units. In particular, it should apply to the very extreme $L_g = 1$, 2 and 3 for which exact values of x_f, N_f and σ_f can be calculated analytically.

For $L_g = 1$, trivially $x_f = 0$, $\sigma_f = 0$, and $N_f/L = 1$. For $L_g = 2$ or $L_g = 3$, given a site on a line x, one can describe all the configurations such that the point belongs to the front, and compute their probability. For example, for $L_g = 2$ all the occupied sites on the line $x = 1$ belong to the front, and a site on the line $x = 0$ will belong to the front if at least one of its three neighbours on the line $x = 1$ is empty. Thus we get in this case, $n_f(1) = 1/2$ and $n_f(0) = 1 - (1/2)^3 = 7/8$. With the same kind of arguments, we can make the computations

for $L_g = 3$ but the geometry of connected sets is more complex as there are more configurations to consider. This is not trivial but can be done exactly [13]. One obtains:

– for $L_g = 1$: $N_f/L = 1$, $x_f = 0$ and $\sigma_f = 0$;

– for $L_g = 2$:

$$N_f/L = \frac{11}{8}, \quad x_f = \frac{4}{11} \quad \text{and} \quad \sigma_f = \frac{2\sqrt{7}}{11};$$

– for $L_g = 3$:

$$N_f/L = \frac{9401}{5832}, \quad x_f = \frac{6966}{9401} \quad \text{and} \quad \sigma_f = \frac{9\sqrt{576049}}{9401}.$$

As will be shown, the numerical results described below verify the GP power laws with such precision that the question arises of the existence of a simple mathematical law extending from $L_g = 1$ to infinity. (The fact that these power laws are valid for large systems has been proven through extensive numerical simulations [7, 12, 13]). To answer this question in the small system limit, we proceed in 2 steps.

First we test these laws numerically for L_g between 4 and 50 by searching the best numerical power laws followed by the width. Considering arbitrary test exponents values α between 1.6 and 1.9, we study σ_f^α as a function of L_g between 4 to 50. For each α value, there is a best line $\sigma_f^\alpha = a_\alpha(L_g + b_\alpha)$ fitting the numerical σ_f^α. The possible contribution of a small b_α, invisible for large systems, is necessary here. In particular, one should remark that for $L_g = 1$ the width is strictly 0 so that some negative value of b_α should be present. In the next step, the mean error $d(\alpha)$, defined by $d(\alpha)^2 = (1/47)\sum_{L_g=4}^{50}(\sigma_f(L_g)^\alpha - a_\alpha(L_g + b_\alpha))^2$, is measured numerically as a function of α. The results are shown in fig. 2. One finds a clear minimum for $\alpha = 1.75$, showing that this value gives the best power law fit. Once the best fit with the empirical data is made (see fig. 3), one has the best values for the parameters a and b: $a = 0.297$ and $b = -1.09$. Note that b should be strictly equal to -1 in order to obtain a null width for the trivial case $L_g = 1$.

Another verification of the extreme GP power laws can be obtained from the study of the front length or of the quantity $(N_f/L)^{7/3}$ as a function of L_g. In fig. 3, the diamonds represent the values of $(N_f/L)^{7/3}$ and the best linear fit has equation $Y = c(L_g + d)$ with $c = 0.845$ and $d = 0.88$. This shows indeed that the exponents $4/7$ and $3/7$ can be used down to *the steepest gradients for which the frontier is no longer fractal*. Note that the horizontal scale is linear.

Once the values of a and b are obtained numerically, one can extrapolate down to $L_g = 1$, 2 and 3. The results, denoted by σ_f (4–50), are given in table I. One observes that the numerical extrapolations correspond to the exact values with a good, but not perfect, precision. Note that we do not specify a confidence interval at this stage since the fit occurs through a power law. It is then difficult to give a confidence interval for the coefficients a and b obtained from a least-square linear regression on the values of $\sigma_f^{7/4}$.

In order to obtain a better control on the numerical precision of a and b we made extensive computations of the two cases $L_g = 4$ and $L_g = 5$ with 100 trials on a length $L = 5{\cdot}10^5$. Doing so, one obtains the mean values and their standard deviation: $\sigma_f(L_g = 4)^{7/4} = 0.8658 \pm 0.0009$ and for $\sigma_f(L_g = 5)^{7/4} = 1.1610 \pm 0.0013$. Thus if we compute the equation of the line $\overline{a}(L_g + \overline{b})$ which interpolates the two points $(4, \sigma_f(4)^{7/4})$ and $(5, \sigma_f(5)^{7/4})$, we obtain $\overline{a} = 0.2952 \pm 0.0022$ and $\overline{b} = -1.066 \pm 0.041$. This last result shows that the value -1 enters the confidence interval of the numerical $\overline{b}$. Given the numerical values for $L_g = 4$ and 5, we can also get extrapolated values of σ_f for $L_g = 1$, 2 and 3 (denoted by σ_f (4–5)), together with their confidence interval

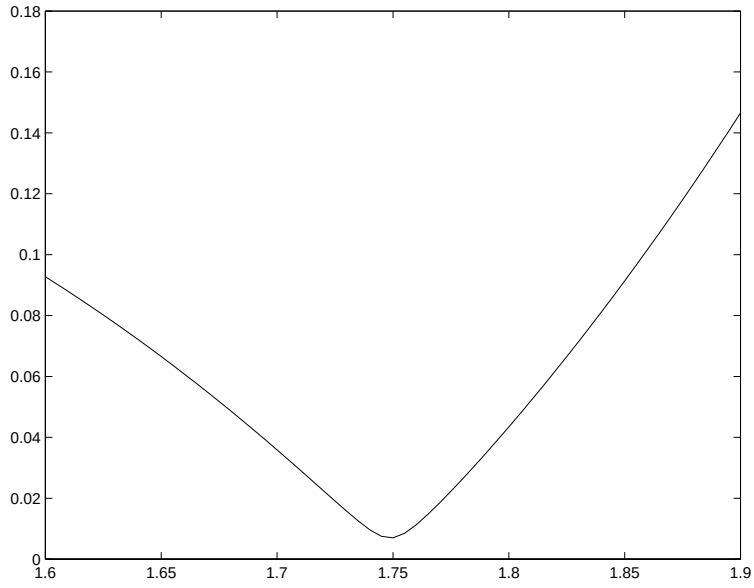

Fig. 2 – Determination of the best exponent value for the square lattice: $\alpha = 1.75$. The same result has been obtained for the triangular lattice [5].

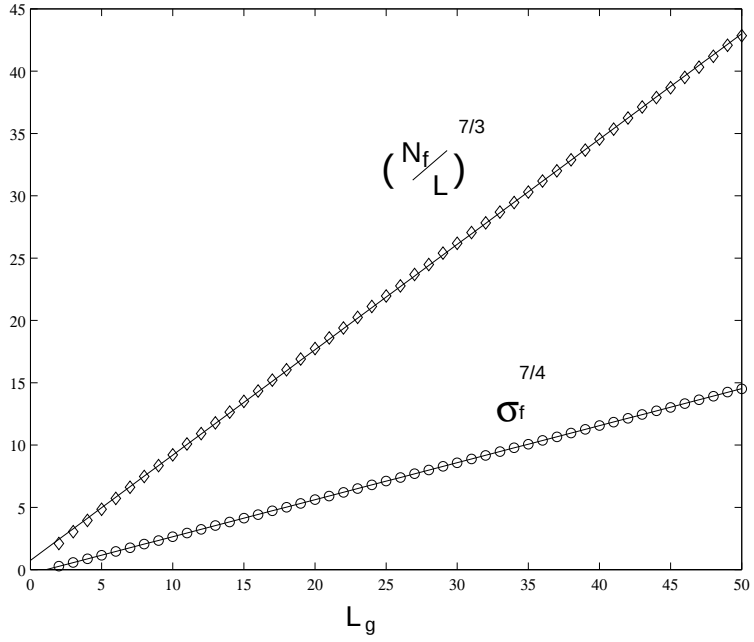

Fig. 3 – Numerical results for the square lattice (data averaged over 50 trials on a length $L = 10^5$). The circles, respectively diamonds, represent, respectively $\sigma_f^{7/4}$ and $(N_f/L)^{7/3}$. Note that scales are linear.

TABLE I – *Comparison between exact and extrapolated results. The data* (4–50)*, respectively* (4–5) *correspond to extrapolated values from the respective ranges* (4–50)*, respectively* (4–5) *(see text).* δ *is the confidence interval.*

	$L_g = 1$	$L_g = 2$	$L_g = 3$
exact σ_f	**0**	**0.4810457**...	**0.7266046**...
σ_f (4–50)	−0.13	0.47	0.72
σ_f (4–5)	0.106	0.478	0.726
$\delta\sigma_f$ (4–5)	0.023	0.005	0.002
exact N_f/L	**1**	**1.3750000**...	**1.6119684**...
N_f/L (4–50)	1.24	1.48	1.68
N_f/L (4–5)	1.109	1.393	1.615
$\delta(N_f/L)$ (4–5)	0.017	0.009	0.004

(denoted by $\delta\sigma_f$ (4–5)). The result (see table I) is that the extrapolated values are very close to the exact ones for $L_g = 2$ and 3. For $(N_f/L)^{7/3}$, in the same way we obtain a linear interpolate of the values for $L_g = 4$ and 5, with coefficients $\overline{c} = 0,893 \pm 0.014$ and $\overline{d} = 0.427 \pm 0.074$. At this point one could conclude that the numerical results are compatible with the existence of a single mathematical law for the width dependence, this law working from $L_g = 2$ to infinity but not for $L_g = 1$. (The quality of the random number generator has been checked by comparing numerical values of σ_f and N_f for $L_g = 2$ and 3, with the exact ones).

But the question of a unique mathematical power law can also be studied from the point of view of the exact results only. As we have exact values, one can compute the equation of the line $y = a(L_g + b)$ defined by the two points $(2, \sigma_f(2)^{7/4})$ and $(3, \sigma_f(3)^{7/4})$. One obtains $a = 0.2939764\ldots$ and $b = -1.0548219\ldots$. This b value also excludes that the power law can be used down to $L_g = 1$. But otherwise, these values enter the confidence interval of the numerics obtained from $Lg = 4$ and 5: $\overline{a} = 0.2952 \pm 0.0022$ and $\overline{b} = -1.066 \pm 0.041$. Thus both numerical and exact results, are compatible with the existence of a unique power law of the form $\sigma_f^{7/4}(L_g) = a(L_g + b)$ working from $L_g = 2$ to infinity.

A few comments can be formulated about the fact that the computed value (supposedly exact if deduced from the $L_g = 2$ and $L_g = 3$ exact values), and the numerical value for b are close, but not equal to -1 (which should be required for the law to fit the case $L_g = 1$). The first remark is that the case $L_g = 2$ is "abnormal" as in this case there is no Grossman-Aharony effect [14]: the accessible perimeter of the front and the front itself are identical. This is not the case for larger L_g values from 3 and above. Secondly the small discrepancy could be related to the fact that the frontier definition considers only the occupied sites. It gives to these sites a privileged role whereas one should also consider the frontier of the empty cluster. In fact this is not new in GP studies [7, 9] where it was shown that the barycenter between the frontier of the occupied cluster and the frontier of the empty cluster was a more natural object. It notably permitted better computations of the percolation threshold. We have then studied the statistical width of the local barycenter (which can also be computed exactly) for $L_g = 2$ or 3. The results show the same behavior as described above, *i.e.* a b value close, but not equal to -1.

The above results have a direct consequence. Given an irregular *non-fractal* interface, for instance the grey line shown in fig. 1, one can determine if it belongs to gradient percolation by measuring its statistical width σ_f and the average value of N_f/L. Through the results described above one can find L_g from the measured σ_f. One can then check if N_f/L satisfies

the GP power laws. This gives an intrinsic way to check if a given interface is of the gradient percolation type *without knowledge* of the gradient. This is important as there exist cases where irregular fronts like corrosion or erosion fronts belong to GP even though the gradient which built the interface is no more present and only the interface remains [12, 15, 16].

In summary, it has been shown that the classical power laws of gradient percolation can be extended to extreme gradients situations with the same fractal exponents although the systems present no fractal geometry. Five conclusions can be drawn from these results. First, we recall that extreme gradient situations can be found in diffused contacts between materials. The fact that the contact geometry can be described by the same exponents whatever the value of the diffusion length will certainly help to understand the properties of these contacts. Second, there is an intrinsic method to find whether a given rough interface belongs to gradient percolation without knowledge of the gradient. Thirdly, our results imply that there exists a (new) conservation law in diffusion. This law stipulates that the length over the correlation length window of the diffusion front is strictly proportional to the gradient or diffusion length. Four, the fact that the same exponents have been found for the square and the triangular lattice even for extreme gradients, in other words for small systems, suggests that universality is not related here to the neglect of the microscopic details of the interactions [13]. Here there is no coarse scale and still universality is verified. Finally, the fact that the exponents 4/7 and 3/7 are valid down to the smallest L_g values (or the steepest gradients) suggests that these exponents play the same type of role here as the exponent 1/2 intervening in the fluctuations of the sum of independent identically distributed random variables. In this last case the exponents apply to *any* number of random variables starting from 2 or 3 up to infinity. The exponents 4/7 and 3/7 may here display a combinatory origin which is out of reach in the thermodynamic limit.

REFERENCES

[1] Stauffer D. and Aharony A., *Introduction to Percolation Theory* (Tailor and Francis, London) 1994.

[2] Sapoval B., Rosso M. and Gouyet J.-F., *J. Phys. Lett. (Paris)*, **46** (1985) L149. The gradient percolation situation has been sometimes called "statistically inhomogeneous random media", bringing confusion with standard percolation which really describes a statistically inhomogeneous random medium in contrast with a deterministic or non-random inhomogeneous medium, like a layered compound, for example.

[3] Sapoval B., Rosso M. and Gouyet J.-F., *The Fractal Approach to Heterogeneous Chemistry*, edited by Avnir D. (John Wiley and Sons Ltd., New York) 1989.

[4] Saleur H. and Duplantier B., *Phys. Rev. Lett.*, **58** (1987) 2325.

[5] Smirnov S., *C. R. Acad. Sci. Paris, Ser. I*, **333** (2001) 239.

[6] Smirnov S. and Werner W., *Math. Res. Lett.*, **8** (2001) 729.

[7] Rosso M., Gouyet J.-F. and Sapoval B., *Phys. Rev. B*, **32** (1985) 6053.

[8] Rosso M., *J. Phys. A*, **22** (1989) L131.

[9] Ziff R. M. and Sapoval B., *J. Phys. A*, **19** (1986) L1169-1172.

[10] Quintanilla J., Torquato S. and Ziff R. M., *J. Phys. A*, **33** (2000) L399.

[11] Quintanilla J., *Phys. Rev. E*, **63** (2001) 061108.

[12] Gabrielli A., Baldassarri A. and Sapoval B., *Phys. Rev. E*, **62** (2000) 3103.

[13] Desolneux A., Sapoval B. and Baldassarri A., `http://arxiv.org/abs/cond-mat/0302072`.

[14] Grossman T. and Aharony A., *J. Phys. A*, **19** (1986) L745.

[15] Balázs L., *Phys. Rev. E*, **54** (1996) 1183.

[16] Sapoval B., Baldassarri A. and Sapoval B., *Phys. Rev. Lett.*, **99** (2004) 098501.